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Symmetries leading to inflation

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Abstract

We present here the general transformation that leaves unchanged the form of the field equations for perfect fluid Friedmann–Robertson–Walker and Bianchi V cosmologies. The symmetries found can be used as algorithms for generating new cosmological models from existing ones. A particular case of the general transformation is used to illustrate the crucial role played by the number of scalar fields in the occurrence of inflation. Related to this, we also study the existence and stability of Bianchi V power law solutions.

I. INTRODUCTION

Symmetries are the cornerstone of the development of modern physics and in particular of the description of the Universe at large scales. Even General Relativity, the framework used for that description, has its roots in symmetry requirements. Not long ago, a new proposal for exploiting the symmetries of the Einstein field equations was made [1]. In this framework two spacetimes are said to be equivalent if the corresponding set of equations is form invariant under the action of a given transformation. This unusual concept of equivalence has been successfully applied to the study of equivalences among different cosmological models, and in particular to those exhibiting inflation.

The equations that govern the evolution of spatially flat Friedmann–Robertson–Walker (FRW) cosmologies filled with a perfect fluid or massive scalar fields happen to admit a peculiar type of form invariance. It directly relates an increase in the energy density of the model with an increase in its expansion rate. This provides a mechanism for using non-accelerating FRW models as seeds for inflationary cosmologies.

Remarkably, this link between the energy density and the expansion rate is the key feature of the assisted inflation proposal. According to it, for some cosmological models, the occurrence of inflation is directly related to the number of scalar fields driving the expansion [2]. This cooperative effect can be easily illustrated in terms of the form invariance we alluded to. In particular, for the expansion rate of a spatially flat FRW model to increase by a factor of n it is only necessary that the energy density gets multiplied by n^2. In the language of scalar fields this just means that a universe containing a single selfinteracting field has transformed into one with n fields interacting with themselves but not amongst them.

The natural question that comes to mind is whether this form invariance symmetry is just a very special feature of spatially flat FRW, rather than commonplace. We have addressed here this question, and our results show that form invariance transformations do exist for any FRW models and their simplest generalization, Bianchi V models.

In sections I and II we outline the details of the transformations of the metric functions, pressure and energy density of perfect fluid FRW and Bianchi V cosmologies leading to unchanged field equations. Then, we exploit the equivalence between that kind of matter content and a scalar field with a self-interaction potential to write equivalent transformation rules for them. The existence of this symmetry provides us with algorithms to generate new cosmological solutions from known ones. We are mainly concerned here by the possibility of generating inflationary solutions from others that do not show that behaviour.

A straightforward particularization of the transformation allows obtaining spacetimes with a Hubble factor that is a constant times the original one. As a consequence, the transformed deceleration factor becomes smaller than the original one, provided this constant is larger than unity. One key feature here is that this information on the spacetime expansion is obtained without knowing the metric functions explicitly.

This is, as we see, very similar to what happened in the assisted inflation scenario (we will dwell on the identification of the analogies in the sections below). Previous studies on this topic have focused on power-law solutions because they are late time attractors for the evolution of both FRW [2] and Bianchi I-VI$_0$ [3] models (see as well [4] for a discussion regarding general geometries). We reach in Section III an analogous result in the Bianchi V case, and in particular, we will see that the expression of the potential in terms of the scale factor for the power-law solutions gives the clue to a method of obtaining new exact scalar field solutions up to quadratures.

The discussion fits into a method to obtain solutions that, as will be explained in Section IV, has turn out to be fruitful before. The scalar field potentials, or the equation of state in the case of perfect fluids, and the evolution of the scale factor are derived from the history of the potential. We use these solutions to present simple examples of the action of the transformation on models with well known potentials. Finally, we outline our future prospects and main conclusions in section V.

II. FORM INVARIANCE SYMMETRY IN FRW SPACETIMES

Let us consider a FRW spacetime (isotropic and spatially homogeneous) with curvature k, which is described by the metric

$$ds^2 = -dt^2 + a^2(t)\left[\left(1 - kr^2\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]. \tag{1}$$

If the source of the geometry is a perfect fluid with energy density ρ and identical pressure p along the three spatial directions (isotropic perfect fluid), then the model is governed by the Friedmann equation

$$3H^2 + 3\frac{k}{a^2} = \rho \tag{2}$$

and the energy conservation equation

$$\dot{\rho} + 3H(\rho + p) = 0, \tag{3}$$

where, $H = \dot{a}/a$ is the Hubble factor, as usual. Given a different perfect fluid with energy density $\bar{\rho}$ and pressure $\bar{p}$, the corresponding equations take the form

$$3\bar{H}^2 + 3\frac{k}{\bar{a}^2} = \bar{\rho}, \tag{4}$$

$$\dot{\bar{\rho}} + 3\bar{H}(\bar{\rho} + \bar{p}) = 0. \tag{5}$$

Our goal is to obtain a transformation that leaves the form of the system of equations (4)–(5) unchanged. In other words, we want to find the symmetry transformation that maps (4)–(5) into (2)–(3). Since we mean to obtain this transformation explicitly, we make the following ansatz:

$$\bar{a} = \bar{a}(a, H, \rho, p), \tag{6}$$

$$\bar{H} = \bar{H}(a, H, \rho, p), \tag{7}$$

$$\bar{\rho} = \bar{\rho}(a, H, \rho, p), \tag{8}$$

$$\bar{p} = \bar{p}(a, H, \rho, p). \tag{9}$$

From equations (2)–(3) and (4)–(5) we obtain the Raychaudhuri equation

$$\dot{H} = -\frac{1}{2}(\rho + p) + \frac{k}{a^2} \tag{10}$$

and its transformed version

$$\dot{\bar{H}} = -\frac{1}{2}(\bar{\rho} + \bar{p}) + \frac{k}{\bar{a}^2}. \tag{11}$$

In what follows, we are going to use the fact that a transformation can be regarded as a symmetry transformation when it does not impose restrictions on the functions appearing in the equations, that is, when one always obtains identities for any function. If we differentiate Eq. (7) a term proportional to $\dot{p}$ arises, but if we insert the expression in the l.h.s. of Eq. (11), it can be noticed immediately that it must vanish identically for there are no terms proportional to $\dot{p}$ in the r.h.s. Then, the first conclusion we draw is that $\bar{H} = \bar{H}(a, H, \rho)$.

If we replace now Eqs. (6) and (8) in Eqs. (4) and (5), and use the same argument we conclude that $\bar{a} = \bar{a}(a, H, \rho)$ and $\bar{\rho} = \bar{\rho}(a, H, \rho)$.

The next step is to calculate $\bar{H}$ from Eq. (6) using the definition $\bar{H} = \dot{\bar{a}}/\bar{a}$, so that

$$\bar{H} = \frac{a}{\bar{a}}\frac{\partial \bar{a}}{\partial a} H + \frac{1}{\bar{a}}\frac{\partial \bar{a}}{\partial H}\left[\frac{k}{a^2} - \frac{1}{2}(\rho + p)\right] - \frac{3H(\rho + p)}{\bar{a}}\frac{\partial \bar{a}}{\partial \rho}, \tag{12}$$

where equations (3) and (10) have been used. Since $\bar{H}$ does not depend on p, the coefficient of p in Eq. (12) must vanish as well. For that reason,

$$\frac{\partial \bar{a}}{\partial (3H^2)} + \frac{\partial \bar{a}}{\partial \rho} = 0, \tag{13}$$

and the general solution to the latter is $\bar{a} = \bar{a}\left(a, \rho - 3H^2\right) = \bar{a}\left(a, 3k/a^2\right)$, that is, $\bar{a}$ depends on a only.

Summarizing, the transformation turns out to be

$$\bar{a} = \bar{a}(a), \tag{14}$$

$$\bar{H} = \frac{\partial \ln \bar{a}}{\partial \ln a} H, \tag{15}$$

$$\bar{\rho} = 3\left[\frac{\partial \ln \bar{a}}{\partial \ln a}\right]^2 H^2 + \frac{3k}{\bar{a}^2}, \tag{16}$$

$$\bar{p} = -\bar{\rho} - 2\dot{\bar{H}} + 2\frac{k}{\bar{a}^2}, \tag{17}$$

where $\dot{\bar{H}}$ has to be calculated using Eq. (15). Here, $\bar{a} = \bar{a}(a)$ is the only "parameter" of the symmetry transformation. Once more, we must emphasize the general character of the transformation as no assumption on the equation of state of the fluid had to be made.

Now, it is interesting to investigate the transformation properties of other physical parameters, so that one can deepen in the comparison between the features of the two models. For instance, the deceleration parameter

$$q(t) = -\frac{\ddot{a}}{aH^2} \tag{18}$$

transforms as

$$\bar{q} + 1 = \left[\frac{\partial \ln \bar{a}}{\partial \ln a}\right]^{-1}(q + 1) + \frac{\partial}{\partial \ln a}\left[\frac{\partial \ln \bar{a}}{\partial \ln a}\right]^{-1} \tag{19}$$

under the symmetry transformations (8)–(9).

As an example let us look at the power–law transformation $\bar{a} = a^n$. In this particular case, we obtain from Eqs. (15) (16) and (17) the transformation rules $\bar{H} = nH$,

$$\bar{\rho} = n^2\left[\rho - 3\frac{k}{a^2}\right] + 3\frac{k}{a^{2n}} \tag{20}$$

and

$$\bar{p} = -3n^2H^2 - 2n\dot{H} - \frac{k}{a^{2n}}. \tag{21}$$

The transformation rule for the deceleration parameter is

$$\bar{q} = -1 + \frac{q+1}{n}. \tag{22}$$

It becomes negative for n high enough, showing in this case that the inflation is more likely. In other words, the strong energy condition (SEC) $\bar{\rho} + 3\bar{p} = -6(n^2H^2 + n\dot{H}) > 0$ is violated for large n. So, we will be interested in the cases with $n > 1$, where the transformed evolution will always be closer to de Sitter spacetime than the original one.

The fluid interpretation of the scalar field has proven very useful in the study of the inflationary and quintessence scenarios [5]. The energy-momentum tensor of the scalar field may be written in the perfect fluid form

$$T_{ik} = (p+\rho)u_i u_k + p g_{ik}, \tag{23}$$

if one defines

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi), \tag{24}$$

$$p = \frac{1}{2}\dot{\phi}^2 - V(\phi), \tag{25}$$

where we have taken into account that the scalar field ϕ depends on t only. Inserting Eqs. (24)–(25) in Eqs. (20)–(21) we find that

$$\dot{\bar{\phi}}^2 = n\dot{\phi}^2 + 2k\left[\frac{1}{a^{2n}} - \frac{n}{a^2}\right], \tag{26}$$

$$\bar{V} = nV + 3n(n-1)H^2 + 2k\left[\frac{1}{a^{2n}} - \frac{n}{a^2}\right]. \tag{27}$$

Asymptotically (i.e., when $a \to \infty$) and for large and positive n the transformation (20) reduces to $\bar{\rho} \approx n^2\rho$ and we recover in this limit the results found in [1] for the particular case of flat FRW metrics. Under these specific conditions one can speak in terms of a single field model transforming into a multifield model in the fashion of assisted inflation.

III. BIANCHI V MODELS

We wish to investigate now the form invariance symmetry of the Einstein equations of an anisotropic universe described by the Bianchi V metric. These spacetimes are the simplest generalizations of FRW open universes and they possess a non-abelian group of three isometries. We write the line element as (cf. [6])

$$ds^2 = -dt^2 + e^{f(t)}dz^2 + G(t)e^z\left(e^{h(t)}\,dx^2 + e^{-h(t)}\,dy^2\right), \tag{28}$$

with $G(t) \geq 0$. Considering once again a universe filled with an isotropic perfect fluid, the Einstein equations for (28) are

$$\frac{\ddot{G}}{G} + \frac{\dot{G}}{2G}\dot{f} + p - \rho - e^{-f} = 0, \tag{29}$$

$$\ddot{h} + \dot{h}\left(\frac{\dot{f}}{2} + \frac{\dot{G}}{G}\right) = 0, \tag{30}$$

$$\frac{\dot{G}}{G} - \dot{f} = 0, \tag{31}$$

$$\frac{\ddot{G}}{G} - \frac{\dot{G}\dot{f}}{2G} - \frac{\dot{G}^2}{2G^2} + \frac{\dot{h}^2}{2} + p + \rho + \frac{e^{-f}}{2} = 0, \tag{32}$$

$$\dot{\rho} + \frac{3\dot{G}}{2G}(\rho + p) = 0. \tag{33}$$

From Eqs. (30) and (31) one readily gets

$$e^f = G, \tag{34}$$

$$\dot{h} = \frac{A}{G^{3/2}}, \tag{35}$$

with the integration constant $A \neq 0$. Combining the remaining Einstein equations in terms of the new variables

$$a = G^{1/2}, \qquad H = \frac{\dot{a}}{a} = \frac{\dot{G}}{2G}, \tag{36}$$

we get

$$3H^2 = \rho + \frac{3}{4a^2} + \frac{A^2}{4a^6}, \tag{37}$$

$$\dot{\rho} + 3H(\rho + p) = 0. \tag{38}$$

These equations are equivalent to those of negatively curved FRW models filled with a perfect fluid and stiff matter. The shear's contribution to the expansion is precisely the last term in Eq. (37) and basically can be associated with a free scalar field. Equation (37) together with the conservation equation (38) forms the set of equations to be solved.

A. Form invariance symmetry

Remarkably, the pair formed by equations (37) and (38) is form-invariant as well. This means that for a different fluid with density $\bar{\rho}$ and pressure $\bar{p}$ the equations

$$3\bar{H}^2 = \bar{\rho} + \frac{3}{4\bar{a}^2} + \frac{\bar{A}^2}{4\bar{a}^6}, \tag{39}$$

$$\dot{\bar{\rho}} + 3\bar{H}(\bar{\rho} + \bar{p}) = 0 \tag{40}$$

become equations (37)–(38) under the symmetry transformation

$$\bar{a} = \bar{a}(a), \tag{41}$$

$$\bar{H} = \frac{\partial \ln \bar{a}}{\partial \ln a} H, \tag{42}$$

$$\bar{\rho} = 3\left[\frac{\partial \ln \bar{a}}{\partial \ln a}\right]^2 H^2 - \frac{3}{4\bar{a}^2} - \frac{\bar{A}^2}{4\bar{a}^6}, \tag{43}$$

$$\bar{p} = -\bar{\rho} - 2\dot{\bar{H}} - \frac{1}{2\bar{a}^2} - \frac{\bar{A}^2}{2\bar{a}^6}, \tag{44}$$

where $\dot{\bar{H}}$ has to be calculated using Eq. (42). Here, $\bar{a} = \bar{a}(a)$ is the "parameter" of the transformation.

Since our main objective is to use this symmetries in the context of inflation, it is of interest to exploit the customary equivalence between a perfect fluid and a self-interacting scalar field with potential $V(\phi)$ when $\bar{a} = a^n$. In this case equations (37)–(38) become

$$3H^2 = \frac{1}{2}\dot{\phi}^2 + V + \frac{3}{4a^2} + \frac{A^2}{4a^6}, \tag{45}$$

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V}{\partial \phi} = 0, \tag{46}$$

where the last one is the Klein-Gordon equation. Transforming Eqs. (24)–(25) according to Eqs. (43)–(44) we obtain the transformation properties of the scalar field and the corresponding potential

$$\dot{\bar{\phi}}^2 = n\dot{\phi}^2 + \frac{1}{2}\left(\frac{n}{a^2} - \frac{1}{a^{2n}}\right) + \frac{1}{2}\left(\frac{nA^2}{a^6} - \frac{\bar{A}^2}{a^{6n}}\right), \tag{47}$$

$$\bar{V} = nV + 3n(n-1)H^2 + \frac{n}{2a^2} - \frac{1}{2a^{2n}}. \tag{48}$$

It can be shown that the transformation rule for the deceleration factor under the transformation $\bar{a} = a^n$ is again Eq. (22).

B. Asymptotic behaviour and power-law solutions

The combined measurements of the cosmic microwave background temperature fluctuations and the distribution of galaxies on large scales seem to imply that the Universe may be flat or nearly flat [7, 8, 9]. The equivalence between the equations (37)–(38) for the Bianchi V model and those of open FRW, permits to introduce the cosmological density parameter Ω, defined as the ratio of the scalar field energy density ρ to the asymptotic critical density $\rho_c = 3H^2$. The asymptotic critical density corresponds to the asymptotic flat spacetime solution of equation (45). Thus, inserting $\Omega \equiv \rho/\rho_c = \rho/3H^2$ in Eqs. (37)–(38), we find the dynamical equation for the cosmological density parameter:

$$\dot{\Omega} = \left[6\frac{a^4 + A^2}{3a^4 + A^2} - 3\left(1 + \frac{p}{\rho}\right)\right]\Omega(1 - \Omega)H. \tag{49}$$

Stable constant solutions of this equation are relevant to understand the asymptotic behaviour of that spacetime, where Eq. (49) reduces to

$$\dot{\Omega} = -\left(1 + 3\frac{p}{\rho}\right)\Omega(1 - \Omega)H + O\left(\frac{1}{a^4}\right). \tag{50}$$

The constant solution of Eqs. (49) and (50), compatible with observations at late times, is $\Omega = 1$. It is asymptotically stable for expanding universes ($H > 0$) when the SEC is violated. Hence the model inflates, i.e., $\ddot{a}/a = -(\rho + 3p)/6 > 0$ and the expansion of the universe is driven by a gravitationally repulsive stress.

The general form of the perfect fluid Bianchi V solution, up to a quadrature, has been known [10] for long (see also ([11], [12],[13]). Here, however, we adopt a different perspective in what exact solutions are regarded. In order to investigate the existence of power-law solutions in the Bianchi V model with a self-interacting scalar field and their stability, we combine the scaling parameter $\omega = -2\dot{H}/3H^2$ with equations (45) and (46), so that we get

$$\dot{\omega} = (\omega - 2)\left[\frac{\dot{V} - (H/a^2)}{V + (1/2a^2)} + 3H\omega\right]. \tag{51}$$

The fixed point solution of Eq. (51), $\omega = \omega_0 =$ constant, corresponds to

$$a = t^{2/3\omega_0}, \tag{52}$$

$$V = \frac{2\,(2 - \omega_0)}{3\omega_0^2 a^{3\omega_0}} - \frac{1}{2a^2}, \tag{53}$$

where the scalar field can be calculated from Eq. (45):

$$\frac{1}{2}\dot{\phi}^2 = \frac{2}{3\omega_0 t^2} - \frac{1}{4t^{4/3\omega_0}} - \frac{A^2}{4t^{4/\omega_0}}. \tag{54}$$

Furthermore, equation (51) becomes

$$\dot{\omega} = 3(\omega - 2)(\omega - \omega_0)H. \tag{55}$$

Therefore, for the potential (53) the power-law solution is asymptotically stable whenever be $\omega_0 < 2$. When $\omega_0 < 2/3$ the SEC is violated and there is an accelerated scenario. If $2/3 < \omega_0 < 2$ we can construct an inflationary model by substituting ω_0/n for ω_0 and taking n large enough, because

$$\bar{a} = t^{2n/3\omega_0} \tag{56}$$

is an asymptotically stable exact power-law solution of Eqs. (45)–(46) for the potential

$$\bar{V} = \left(2 - \frac{\omega_0}{n}\right)\frac{2n^2}{3\omega_0^2 \bar{a}^{3\omega_0/n}} - \frac{1}{2\bar{a}^2}, \tag{57}$$

where the scalar field can be calculated from Eq. (45):

$$\frac{1}{2}\dot{\bar{\phi}}^2 = \frac{2n}{3\omega_0 t^2} - \frac{1}{4t^{4n/3\omega_0}} - \frac{A^2}{4t^{4n/\omega_0}}. \tag{58}$$

Quantities $(\dot{\phi}, V)$ and $(\dot{\bar{\phi}}, \bar{V})$ are related by transformations (47)–(48). Also from Eqs. (41), (42) and (56), we obtain

$$\bar{a} = a^n, \qquad \bar{H} = nH. \tag{59}$$

For large n the energy density of the scalar field transforms as $\bar{\rho} \approx n^2 \rho$ showing that the cumulative effects of adding energy density in the 00-component of the Einstein equations leads to an accelerated scenario. In this regime $\bar{\phi} \approx 2(n/3\omega_0)^{1/2} \ln t$ and the potential is exponential

$$\bar{V}(\phi) \approx \frac{2n^2}{3\omega_0^2}\left(2 - \frac{\omega_0}{n}\right) \mathrm{e}^{-\sqrt{3\omega_0/n}\,\phi}. \tag{60}$$

When n is an integer it could represent the number of scalar fields contained in the Bianchi V universe. This particular multifield Bianchi V problem is equivalent to realize the usual assisted inflation in the FRW model with n identical non interacting scalar fields, in a negatively curved space-time filled with an extra free scalar field. Hence we have extended the previous results obtained in [1] for FRW to the Bianchi V type metrics. Again we have seen that form invariance of the Einstein equations leads to a simple generalization of the assisted inflation linking two different cosmological models, one of which, is accelerated.

IV. TRANSFORMATION BETWEEN CLOSED FORM SOLUTIONS

We seek now models for which the potential $V(\phi)$ and the scale factor evolution $a(t)$ can be obtained in closed form and exhibit the action of the power–law transformation in simple terms. The expression (57) for the potential suggests to start looking for exact solutions where the scale factor is the independent variable. In Ref. [14] it was shown the reduction of the system (2)(3) to quadratures using as input the history of the potential. In this case Eq. (3) becomes (46), we write the potential as

$$V[\phi(a)] = \frac{F(a)}{a^6} \tag{61}$$

and make the change of variables $dt = a^3 d\eta$ in Eq. (46),

$$\frac{d^2\phi}{d\eta^2} + a^6 \frac{dV}{d\phi} = 0, \tag{62}$$

so that we obtain the first integral

$$\frac{1}{2}\dot{\phi}^2 + V(\phi) - \frac{6}{a^6}\int da \frac{F}{a} = \frac{C}{a^6}, \tag{63}$$

where C is an arbitrary integration constant. Then, using equations (2) and (46) we obtain

$$\Delta t = \sqrt{3}\int \frac{da}{a}\left[\frac{6}{a^6}\int da\frac{F}{a} + \frac{C}{a^6} - 3\frac{k}{a^2}\right]^{-1/2}, \tag{64}$$

$$\Delta\phi = \sqrt{6}\int \frac{da}{a}\left[\frac{-F + 6\int daF/a + C}{6\int daF/a + C - 3ka^4}\right]^{1/2}, \tag{65}$$

where $\Delta t \equiv t - t_0$, $\Delta\phi \equiv \phi - \phi_0$ and t_0, ϕ_0 are two other arbitrary integration constants.

This unconventional procedure of solving the field equations starting from the history of the potential was also used to obtain exact solutions in two and four dimensional spacetimes with a scalar field and a perfect fluid in Refs. [14, 15]. An interesting simple class of models arise for potentials with power–law histories $F = Ba^m$ and $C = 0$. In this case, $\dot{\rho}_\phi = -3H\dot{\phi}^2 < 0$ implies $m < 6$ for $H > 0$, $\dot{\phi}^2 > 0$ implies $B/m > 0$. Then, for positive definite potentials, hence $B > 0$, we have $0 < m < 6$. We find hyperbolic potentials for $m \neq 4$ and $k = 1$

$$V(\phi) = \left(\frac{m}{2}\right)^{\frac{6-m}{4-m}} B^{2/(m-4)}\left\{\cosh^2\left[\frac{(4-m)\Delta\phi}{2(6-m)^{1/2}}\right]\right\}^{\frac{6-m}{4-m}}, \tag{66}$$

or $k=-1$

$$V(\phi)=\left(\frac{m}{2}\right)^{\frac{6-m}{4-m}}B^{2/(m-4)}\left\{\sinh^2\left[\frac{(4-m)\Delta\phi}{2(6-m)^{1/2}}\right]\right\}^{\frac{6-m}{4-m}}, \tag{67}$$

while an exponential potential arise for $m=4$, $k=-1$ or $k=1$ with $B>2$,

$$V(\phi)=B\exp\left[-\left(2\frac{B-2k}{B}\right)^{1/2}\Delta\phi\right]. \tag{68}$$

As we see, potentials of this class include hyperbolic potentials that are relevant to describe the scalar dark matter [16, 17, 18]; and they also include the exponential potential, which is typically considered when modelling inflation, and is motivated by supergravity theories [19, 20].

For these models, the scale factor can be obtained in closed, implicit form when $m\neq 4$ in terms of the hypergeometric function. For $k=1$ we obtain

$$\Delta t=\left(\frac{2m}{B}\right)^{1/2}\frac{a^{3-m/2}}{6-m}\,{}_2F_1\left(\frac{1}{2},\frac{m-6}{2(m-4)},\frac{3m-14}{2(m-4)},\frac{ma^{4-m}}{2B}\right), \tag{69}$$

while for $k=-1$ we get

$$\Delta t=a\,{}_2F_1\left(\frac{1}{2},\frac{1}{m-4},\frac{m-3}{m-4},-\frac{2B}{m}a^{m-4}\right). \tag{70}$$

We wish to see the action of the symmetry transformation on this class of models, linking non-accelerated expansions with accelerated ones. So we start looking at their late time behavior.

For $k=1$ and $4<m<6$, the scale factor has a bounce and its asymptotic behavior for large times is $t^{2/(6-m)}$; while for $0<m<4$, the scale factor has an upper bound so that we will not consider this case any further. For $k=-1$ and $0<m<4$ the scale factor evolves from a unaccelerated initial phase with $a\simeq\Delta t^{2/(6-m)}$ at small times to a linear expansion at late times; while for $4<m<6$ the scale factor evolves from a linear stage at small times to an accelerated stage $a\simeq t^{2/(6-m)}$ at late times. Finally, for $k=\pm 1$ and $m=4$, the expansion is linear.

Taking into account Eq. (63) we observe that the action of the symmetry transformation (14)–(17) becomes the map $(F(a),C)\to(\bar F(\bar a),\bar C)$ where Eq. (64) remains invariant. That is, the set of the solutions that can be expressed in terms of quadratures transforms into itself. In the case of the power–law transformation (26)–(27) we get

$$\bar F=na^{6(n-1)}\left[F+(n-1)\left(6\int\frac{da}{a}F+C\right)\right]-n\left(3n-1\right)ka^{6n-2}+2ka^{4n}. \tag{71}$$

For the models with power–law history of their potential the transformation becomes

$$\bar{F} = nB\left(1 + 6\frac{n-1}{m}\right)a^{6(n-1)+m} - n(3n-1)ka^{6n-2} + 2ka^{4n}. \tag{72}$$

So, the requirement that the transformed potential has a power–law history, i.e. $\bar{F} = \bar{B}\bar{a}^{\bar{m}}$ can be satisfied for $k = -1$ with $\bar{B} = n(3n-1)$ and $\bar{m} = (32-6m)/(6-m) = 6 - 2/n$. The requirements that $m > 0$ and $\bar{m} > 0$ constraint the transformation exponent to $1/3 < n < 3$ and the history exponents to the interval $(0, 16/3)$. We observe that the transformation with $n > 1$ maps unaccelerated models with $m < 4$ into accelerated ones with $\bar{m} > 4$, while the linearly expanding model with $m = 4$ transforms into itself.

As a second example we consider transformations for $k = \pm 1$ that allow an arbitrary large exponent. In this case we will request that the transformed potential has power–law history only asymptotically for $n \gg 1$ and large scale factor. This case occurs for $m = 4$ and we have

$$\bar{B} = n\left\{B\left[1 + \frac{3}{2}(n-1)\right] - (3n-1)k\right\} \tag{73}$$

and again $\bar{m} = 6 - 2/n$. This transformation maps the linearly expanding model with exponential potential into a model with deceleration parameter so close to -1 as required while n is made large enough. Equations (66) (67) show explicitly how the slope of the potential can be made arbitrary small in this limit.

Following similar steps we can integrate the system (45)–(46) by quadratures.

$$\Delta t = \sqrt{3}\int\frac{da}{a}\left[\frac{6}{a^6}\int da\frac{F}{a} + \left(C + \frac{A^2}{4}\right)\frac{1}{a^6} + \frac{3}{4a^2}\right]^{-1/2}, \tag{74}$$

$$\Delta\phi = \sqrt{6}\int\frac{da}{a}\left[\frac{-F + 6\int daF/a + C}{6\int daF/a + C + A^2/4 + 3a^4/4}\right]^{1/2}. \tag{75}$$

In this case, the action of the symmetry transformation (41)–(44) becomes the map $(F(a), C, A) \to (\bar{F}(\bar{a}), \bar{C}, \bar{A})$ where Eq. (74) remains invariant. In the case of the power–law transformation (47)–(48) we get

$$\bar{F} = na^{6(n-1)}\left[F + (n-1)\left(6\int\frac{da}{a}F + C + \frac{A^2}{4}\right)\right] + \frac{n\,(3n-1)}{4}a^{6n-2} - \frac{a^{4n}}{2} \tag{76}$$

and for models with power–law history of their potential it becomes

$$\bar{F} = nB\left(1 + 6\frac{n-1}{m}\right)a^{6(n-1)+m} + \frac{n(3n-1)}{4}a^{6n-2} - \frac{a^{4n}}{2}. \tag{77}$$

Comparing Eqs. (74) and (75) with Eqs. (64) (65) we see that the solutions of FRW for $k = -1$ give the leading behavior of the solutions of Bianchi V spacetime. Hence, the simple action of the power law transformation in FRW still holds for Bianchi V in the asymptotic sense.

We recover the solution (52)–(54) inserting

$$F = \frac{2(2-\omega_0)}{3\omega_0^2} a^{3(2-\omega_0)} - \frac{a^4}{2} \tag{78}$$

into Eqs. (61), (74) and (75), and similarly we recover the asymptotic behaviour (60) inserting into Eq. (75) the leading term of (78), transformed by (76), when $\omega_0 < 2n/3$.

V. CONCLUSIONS

We have shown the symmetry transformation under which the Einstein equations in the general Friedmann–Robertson–Walker and Bianchi V cosmologies are form invariant. It relates geometrical quantities with the energy density and pressure of the perfect fluid. We have seen that the cooperative effect of adding energy density into the spacetime leads to inflation and gives a way to link a non-accelerated scenario with an inflationary scenario by means of a symmetry transformation.

As an example, we have investigated the connections between assisted inflation and this symmetry transformation. We have shown that assisted inflation can be generalized to any potential that increases the energy density of the scalar field configuration without specifying the number of fields.

The violation of the strong energy condition has been shown to be required for the asymptotic stability of power-law solutions in Bianchi V spacetime, and the potential leading to these solutions has been found.

We have also constructed the invariant set of perfect fluid Friedmann–Robertson–Walker and Bianchi V spacetimes that may be obtained by quadratures. These solutions are characterized by the history of the scalar field potential and their parametric expression is shown. We give the action of the symmetry transformation in this representation.

Finally, we conclude that it is very interesting to study this kind of symmetry transformations, which have received up to now little attention. We shall continue exploring this subject for other metrics in future papers.

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